Math 105 Chapter 12: Power Series

So far we have been interested in adding up numbers. But this is calculus, we need functions to play with. Let us now add functions, and define a function based on series.

Definition: A power series centered at a is a function of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

The c_n's are called the coefficients of the series.

a is called the center of the series.

eg $\sum_{n=0}^{\infty} x^n$, $a=0$, $c_n = 1$

$\sum_{n=0}^{\infty} \frac{(x-1)^2}{n!}$, $a = 1$, $c_n = \frac{1}{n!}$

$\sum_{n=2}^{\infty} \frac{(-3)^n}{n^2}(x+5)$, $a=-5$, $c_0 = c_1 = 0$, $c_n = \frac{(-3)^n}{n^2}$, $n \geq 2$.

Now we already know that series don't always converge so which values of x does $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ make sense?

ie. What is the domain of f?

Well $f(x)$ always makes sense at $x=a$, since

$$f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + c_3(a-a)^3 + \cdots$$
$$= c_0$$

eg $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

$x=0 \Rightarrow f(0) = 1$

$x = \frac{1}{2} \Rightarrow f(\frac{1}{2}) = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$
$$= 2$$

$x = 2 \Rightarrow f(2) = 1 + 2 + 4 + \cdots$
$$= \infty$$

So $f(2)$ does not make sense.

In general $f(x)$ is a geometric series with ratio x. So it converges only when $|x|<1$, and diverges when $|x|>1$.

Our next theorem generalizes the above example.

Theorem: For a given power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

there are 3 possibilities.

1) The series only converges at $x=a$

2) There is a $R>0$ such that when

- $|x-a|<R$, the series converges absolutely
- $|x-a|>R$, the series diverges

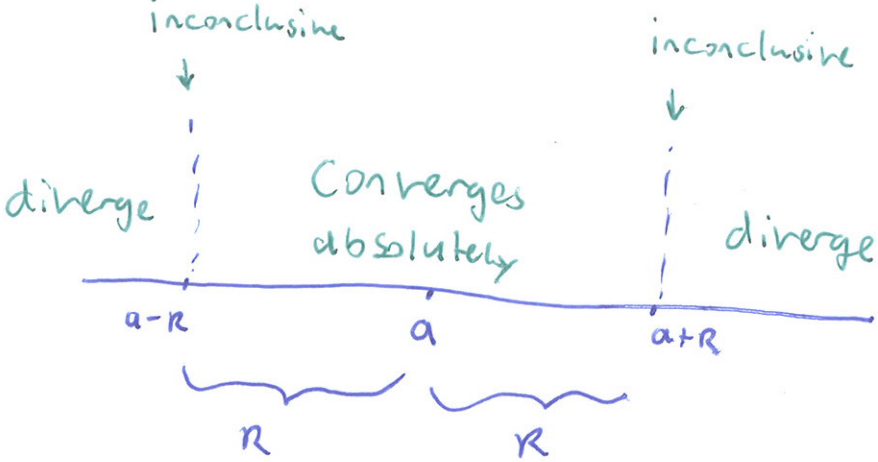
3) The series converges absolutely for all $x \in \mathbb{R}$.

We call the R in the above theorem the radius of convergence.

In case 1) we say $R=0$

3) we say $R=\infty$.

The following diagram sumarizes the theorem:



In general we find the radius of convergence by the ratio test

eg $\sum_{n=0}^{\infty} x^n$, $a=0$, $c_n=1$

$$\lim_{n\to\infty} \left| x^{n+1} / x^n \right| = \lim_{n\to\infty} |x|$$

$$= |x|$$

The series converges absolutely when $|x|<1$ and diverges when $|x|>1$ by ratio test. Thus the radius of convergence is 1, as expected.

eg $\sum_{n=1}^{\infty} \frac{(-3)^n (x-1)^n}{n^2}$, $a=1$, $c_n = \frac{(-3)^n}{n^2}$, $n \geq 1$, $c_0 = 0$

$$\lim_{n\to\infty} \left| \frac{(-3)^{n+1}(x-1)^{n+1}}{(n+1)^2} \bigg/ \frac{(-3)^n (x-1)^n}{n^2} \right|$$

$$= \lim_{n\to\infty} \frac{3^{n+1} |x-1|^{n+1} n^2}{3^n |x-1|^n (n+1)^2}$$

$$= \lim_{n\to\infty} 3|x-1| \frac{n^2}{(n+1)^2}$$

$$= 3|x-1|$$

$3|x-1| < 1 \Rightarrow |x-1| < \frac{1}{3}$

Thus the radius of convergence is $\frac{1}{3}$.

eg $\sum_{n=3}^{\infty} n!\,(x+5)^n$, $a=-5$, $c_n = \begin{cases} 0 & , n \le 2 \\ n! & , n \ge 3 \end{cases}$

When $x \neq -5$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x+5)^{n+1}}{n!\,(x+5)^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!} |x+5|$$

$$= \lim_{n\to\infty} (n+1)\,|x+5|$$

$= \infty$, for all $x \neq -5$

> 1 for all $x \neq -5$

Thus the radius of convergence is 0.

eg $\sum_{n=0}^{\infty} \frac{(x+5)^n}{n!}$, $a=-5$, $c_n = \frac{1}{n!}$

$$\lim_{n\to\infty} \left| \frac{(x+5)^{n+1}}{(n+1)!} \Big/ \frac{(x+5)^n}{n!} \right|$$

$$= \lim_{n\to\infty} |x+5| \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{|x+5|}{(n+1)}$$

$= 0$ for all x

< 1 for all x

So radius of convergence is ∞.

eg $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$, $a=0$, $c_n = \begin{cases} 0 & , n=0 \\ \frac{n!}{n^n} & , n \geq 1 \end{cases}$

$$\lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} x^{n+1} \Big/ \frac{n!\, x^n}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!\, n^n\, x^{n+1}}{n! \quad (n+1)^{n+1}\, x^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)\, n!\, n^n |x|}{n!\, (n+1)(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} |x|$$

$\left[\text{fact: } e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \right]$

$$= \lim_{n\to\infty} \left[\left(\frac{n+1}{n} \right)^n \right]^{-1} |x|$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{n} \right)^n \right]^{-1} |x|$$

$$= e^{-1} |x|$$

$e^{-1}|x| < 1 \implies |x| < e$

So the radius of convergence is e.

So far we have been looking at the domain of

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

But we want a way to determine what $f(x)$ actually is without a sum.

eg Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Let us find a formula for $f(x)$. Let's see what we get when I take the derivative.

$$f'(x) = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= f(x)$$

So $y = f(x)$ satisfies the differential equation.

$$\frac{dy}{dx} = y$$

Let's solve for y.

$$\int \frac{dy}{y} = \int dx$$

$$\log|y| = x + C$$

So we have

$$|y| = e^c e^x$$

$$y = \pm e^c e^x$$

$$= Ae^x \qquad , A = \pm e^c$$

Now to solve for A, we note

$$f(0) = 1 + 0 + \frac{0^2}{2!} + \frac{0^3}{3!} + \cdots$$

$$= 1$$

So $x=0$ implies $y=1$ and thus $A=1$, and thus we have shown.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Now as you can imagine, we can't always find a differential equation our power series will solve. We need a better way.

Let's suppose

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

We will find a formula for c_n. Notice when $x=a$ we have

$$f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots$$

$$= c_0$$

If we differentiate our function, we get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

Letting $x=a$, we get.

$$f'(a) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 \cdots$$
$$= c_1$$

Let's differentiate again.

$$f''(x) = 2c_2 + 3\cdot 2\, c_3(x-a) + 4\cdot 3\, c_4(x-a)^2 + 5\cdot 4\, c_5(x-a)^3 + \cdots$$

Letting $x=a$, we get

$$f''(a) = 2c_2 \Rightarrow c_2 = \frac{f''(a)}{2}$$

Again!

$$f'''(x) = 3\cdot 2\cdot c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + 5\cdot 4\cdot 3(x-a)^2 + \cdots$$

Let $x=a$

$$f'''(a) = 3\cdot 2\cdot c_3 \Rightarrow c_3 = \frac{f'''(a)}{3\cdot 2\cdot 1} = \frac{f'''(a)}{3!}$$

One more time!

$$f^{(4)}(x) = 4\cdot 3\cdot 2\cdot c_4 + 5\cdot 4\cdot 3\cdot 2(x-a) + 6\cdot 5\cdot 4\cdot 3(x-a)^2 + \cdots$$

$x=a$

$$f^{(4)}(a) = 4\cdot 3\cdot 2\cdot c_4 \Rightarrow c_4 = \frac{f^{(4)}(a)}{4\cdot 3\cdot 2} = \frac{f^{(4)}(a)}{4!}$$

Hopefully at this point you see a pattern.

$$c_n = \frac{f^{(n)}(a)}{n!}$$

So in general

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Definition A Taylor series of f centered at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

When $a=0$ we call the Taylor Series a Maclaurin Series.

eg Let's find the maclaurin series of $\sin x$.

$$\left.\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ &\vdots \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f^{(4)}(0) &= 0 \\ &\vdots \end{aligned}$$

Thus

$$\sin x = 0 + x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Let's find the radius of convergence, apply ratio test.

$$\left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \Big/ \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right|$$

$$= \frac{(2n+1)!}{(2n+3)!} \frac{|x|^{2n+3}}{|x|^{2n+1}}$$

$$= \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} |x|^2$$

$$= \frac{|x|^2}{(2n+3)(2n+2)}$$

$$\to 0 \quad \text{as } n \to \infty \quad (\text{for all } x).$$

Thus the radius of convergence is ∞.

The next theorem will let us differentiate/integrate power series.

Theorem: If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, $|x-a| < R$, then:

1. $f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^n$, $|x-a| < R$
2. $\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$, $|x-a| < R$

Remarks:

1) the theorem tells us that the linearity of the integral/derivative holds even for infinite sums. ie,

$$\frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx} (x-a)^n \left(= \sum_{n=0}^{\infty} n c_n (x-a)^{n-1} \right)$$

$$\int \sum_{n=0}^{\infty} c_n (x-a)^n dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n dx \left(= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} \right)$$

2) When you differentiate/integrate, the radius of convergence does <u>not</u> change.

Let's use the theorem to find the power series of $\cos x$.

eg Find the Maclaurin series of $\cos x$.

Recall the Macdaddy series of $\sin x$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad |x| < \infty$$

Let's differentiate.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!}, \quad |x| < \infty$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad |x| < \infty$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

We will now use the above theorem to find the power series of more complicated functions.

eg Find taylor series of $\frac{1}{(1-x)^2}$ centered at 0.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Differentiate both sides

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}, \quad |x| < 1$$

eg Find the big mac series of $\log(1+x)$.

First note that

$$\frac{d}{dx}\log(1+x) = \frac{1}{1+x}$$

So if we find the series of $\frac{1}{1+x}$ we can just integrate.

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} \\ &= \sum_{n=0}^{\infty} (-x)^n \quad , \quad |-x| < 1 \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \quad , \quad |x| < 1 \end{aligned}$$

So

$$\begin{aligned} \log|1+x| &= \int \frac{1}{1+x}\, dx \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \quad , \quad |x| < 1 \end{aligned}$$

$x=0$ implies

$$C = \log 1 = 0$$

Thus

$$\begin{aligned} \log(1+x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \quad , \quad |x| < 1 \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \end{aligned}$$

One can show that the series converges at $x=1$ as well so

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \log 2$$

eg Find maclaurin series of $\frac{x}{2+x}$

Let us first find the power series of $\frac{1}{2+x}$

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})}$$

$$= \frac{1}{2}\,\frac{1}{1-(-\frac{x}{2})}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n, \quad \left|-\frac{x}{2}\right|<1$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n}, \quad |x|<2$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$$

So by multiplying both sides by x we get

$$\frac{x}{2+x} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{2^{n+1}}, \quad |x|<2$$

$n+1=k \rightarrow$ $$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1} x^k}{2^k}, \quad |x|<2,$$

eg Find the mac daddy series of $\frac{x^3}{(1+x^2)^2}$

First let's find the series of $\frac{1}{1+x}$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

$$= \sum_{n=0}^{\infty} (-x)^n \quad , \quad |-x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n \quad , \quad |x| < 1$$

By differentiating both sides

$$\frac{-1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n n x^{n-1} \quad , \quad |x| < 1$$

Replace x with x^2 we get

$$\frac{-1}{(1+x^2)^2} = \sum_{n=0}^{\infty} (-1)^n n (x^2)^{n-1} , \quad |x^2| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n n x^{2n-2} , \quad |x| < 1$$

multiplying both sides by $-x^3$

$$\frac{x^3}{(1+x^2)^2} = -x^3 \sum_{n=0}^{\infty} (-1)^n n x^{2n-2} , \quad |x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} n x^{2n+1} , \quad |x| < 1$$

eg Find the macintosh series of $\arctan x$.

Note $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$, so let's find the power series of $\frac{1}{1+x^2}$.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n \quad , \quad |-x^2| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad , \quad |x| < 1$$

By integrating both sides

$$\arctan x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad , \quad |x| < 1$$

Letting $x=0$ we get

$$C = \arctan 0 = 0$$

Thus
$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad , \quad |x| < 1$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Again one can show that the series converges at $x=1$ so

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

This gives a formula for π (OMG!)

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

We can even use power series to find integrals of functions we previously couldn't.

eg Find $\int e^{x^2} dx$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \quad |x| < \infty$$

so

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} \quad , \quad |x| < \infty$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \quad , \quad |x| < \infty$$

By integrating both sides,

$$\int e^{x^2} dx = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\, n!} \quad , \quad |x| < \infty$$

$$= C + x + \frac{x^3}{3 \cdot 2!} + \frac{x^5}{5 \cdot 3!} + \frac{x^7}{7 \cdot 4!} + \cdots$$

for all $x \in \mathbb{R}$! ← not a factorial just me being excited.

eg Find the taylor series of $\frac{1}{1-x}$, centered at $x=3$.

$$\frac{1}{1-x} = \frac{1}{1-(x-3+3)}$$

$$= \frac{1}{1-(x-3)-3}$$

$$= \frac{1}{-2-(x-3)}$$

$$= \frac{-1}{-2} \frac{1}{\left(1+\frac{(x-3)}{2}\right)}$$

$$= \frac{-1}{2} \frac{1}{1-\left[\frac{-(x-3)}{2}\right]}$$

$$= \frac{-1}{2} \sum_{n=0}^{\infty} \left(\frac{-(x-3)}{2}\right)^n, \quad \left|\frac{-(x-3)}{2}\right| < 1$$

$$= \frac{-1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{2^n}, \quad |x-3| < 2$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{2^{n+1}}$$

And with that, we are done! Good job! High five!

Hope you enjoyed reading these as much as I did writing, which shouldn't be too hard since writing sucks.

"In mathematics you don't understand things you just get used to them"

- John Von Neumann